Modern Algebra I: Group Theory

Classwork 3-28

Task 1

A **homomorphism** from a group G to a group H is a mapping that preserves the group operation, i.e. $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in G$ (notice that the only distinction between an isomorphism and a homomorphism is the lack of the bijectivity condition).

Using the definition, prove that (a) one of these is not a function, (b) one is a function but not a homomorphism, and (c) one is a homomorphism.

- $\theta_1 : \mathbb{Z} \to C_4$ given by $\theta_1(n) = R^{2n}$ for all $n \in \mathbb{Z}$
- $\theta_2 : C_4 \to \mathbb{Z}$ given by $\theta_2(R^n) = n$ for all $R^n \in C_4$
- $\theta_3 : C_4 \to \mathbb{Z}$ given by $\theta_3(R^n) = 13$ for all $R^n \in C_4$

Task 2

New Terms! Since a homomorphism may not be 1-1, there may be more than one thing that maps to the identity. *The set of things that map to the identity is called the Kernel*.

Definition: The **kernel** of ϕ to be the set $\ker(\phi) = \{g \in G: \phi(g) = e_h\}$, where e_h is the identity of H.

Since a homomorphism may not be onto, there may be some things in the codomain that don't get hit. The set of things that do get hit is called the Image.

Define the **image** of ϕ to be the set $\mathrm{im}(\phi) = \{h \in H: \phi(g) = h \text{ for some } g \in G\}$.

Find the Kernel and Image of θ_1 above.

Task 3

Let G, H be groups and $\varphi : G \rightarrow H$ be a homomorphism. Prove that the Image and Kernel of φ are subgroups.

Task 4

Suppose $\phi: G \to H$ is a homomorphism. Develop conjectures that answer the following questions (then prove your conjectures!). For some, it may help to test these conjectures with $G = D_3$ and $H = C_6$.

a. Does a homomorphism necessarily map e_G to e_H?

b. If G is abelian, does this force H to be abelian?

c. Can a self-inverse in H be *hit by* something other than a self-inverse from G?

d. Does a self-inverse in G have to be mapped to something other than a self-inverse in H?

e. Does a homomorphism necessarily preserve the order of elements? For example, is it possible for a homomorphism where an order 6 element maps to an order 3 element?