Modern Algebra I: Group Theory Homework 10

46. Partition the group D_4 into subsets of order 2 that form a group under subset multiplication, and prove that this is indeed a group by using an operation table. Then write this new (quotient) group in coset notation, verifying that the left and right cosets are indeed the same. Lastly, prove again that it is a group (this time using coset representative multiplication in an operation table).

47. Construct all of the non-trivial quotient groups for C_8, the rotations of a regular octagon. For each one, list the elements of the quotient group and provide a table of the quotient group.

48. Construct a quotient group of ($\mathbb{Z}$, +) that has 4 elements. As above, list the elements and provide a table. (Is this quotient group isomorphic to the 4-element quotient group we found for D_4?)

49. Write out the elements of $\mathbb{Z}_{24}/\langle 8\rangle$ in coset notation. What is the order of the element $14 + \langle 8\rangle$?

50. Prove **Lagrange's Theorem:** The order of a subgroup divides the order of the ambient group. That is, if H is a subgroup of G, then $|H|$ divides $|G|$.

51. Use Lagrange's Theorem to deduce that the order of each group element divides the order of the group. That is, for all $g \in G$, $|g|$ divides $|G|$. (Hint: use $<g>$).

52. Let G and H be groups and $\phi: G \to H$ a homomorphism. Prove that ϕ is injective if and only if $\ker \phi = \{e_G\}$.

53. Let G be a group and H a normal subgroup of G.
(1) Prove: If G is abelian, then G/H is abelian.
(2) Prove: If G is cyclic, then G/H is cyclic.

54. Prove that the following groups are isomorphic (using the Fundamental Homomorphism Theorem):
(1) $\mathbb{Z}/\langle n\rangle \cong \mathbb{Z}_n$
(2) $D_4/Z(D_4) \cong V_4$
(3) $D_4/C_4 \cong \{\pm 1\}$

55. For each of the homomorphisms in homework problem #43, use the Fundamental Homomorphism Theorem to "recover" or "construct" an isomorphism.

56. This exercise will explicitly illustrate the first isomorphism theorem (fundamental homomorphism theorem). Consider a homomorphism $\theta: D_3 \to C_6$ defined as follows (suggested but not required exercise: verify that this is indeed a homomorphism):

$\theta(e_G) = e_H$ $\theta(R) = e_H$ $\theta(R^2) = e_H$ $\theta(F) = R^3$ $\theta(FR) = R^3$ $\theta(FR^2) = R^3$

a. Find and construct an operation table for the image of this homomorphism, $im\ \theta$.
b. Find the kernel of this homomorphism, $\ker \theta$.

c. Construct and describe the quotient group $D_3 / \ker \theta$ (that is, explicitly list the elements and make an operation table).
d. Verify that $G / \ker \theta \cong \mathrm{im}(\theta)$. (*Hint*: since these are small finite groups, it may be helpful to compare operation tables.)