Name:____________________________

Modern Algebra I: Group Theory Final Exam

1. Let G be a group and H a subgroup of G. Recall that $aH = \{ah: h \in H\}$ denotes an arbitrary left coset of H in G for any $a \in G$.
   (1) Prove that all cosets of H are the same size: $|aH| = |H|$ for any $a \in G$.
   (2) Prove that distinct cosets are disjoint: if $aH \neq bH$, then $aH \cap bH = \emptyset$.
   (3) Parts (1), (2), and (3) (along with the fact that every element $a \in G$ is contained in a coset aH) prove that G is a disjoint union of its H cosets, that is:
   $$G = \coprod_{a \in G} aH$$
   Use these results to prove **Lagrange's theorem:** if G is a finite group and H is a subgroup of G, then $|H|$ divides $|G|$.

2. Let G and H be groups and $\phi: G \to H$ be a group homomorphism.
   (1) Prove that $\phi(e_G) = e_H$
   (2) Prove that $\phi(a)^{-1} = \phi(a^{-1})$ for any $a \in G$
   (3) Prove that $\ker \phi$ is a subgroup of G.
   (4) Prove that $\operatorname{im} \phi$ is a subgroup of H.
   (5) Prove that ϕ is injective if and only if $\ker \phi = \{e_G\}$.

3. Let G be a group, H a subgroup of G.
   (1) Prove that $\{e, R^2\}$ is a normal subgroup of D_4.
   (2) Use an operation table (using either subset multiplication or coset representation multiplication) to prove that $D_4/\{e, R^2\}$ is indeed a group. Prove that this group is isomorphic to another group we know.
   (3) Prove the **normal subgroup test:** H is a normal subgroup of G if and only if $aHa^{-1} \subseteq H$ for any $a \in G$.
   (4) Prove that $Z(G)$ is a normal subgroup of G.
   (5) Let $\phi: G \to H$ be a homomorphism. Prove that $\ker \phi$ is a normal subgroup of G.

4. Let $(G,\cdot)$ and $(H,*)$ be groups and $\phi: G \to H$ an isomorphism:
   (1) Prove: if G is abelian, then H is abelian.
   (2) Prove: if G is cyclic, then H is cyclic.

5. Let G be a group and H a normal subgroup of G.
   (1) Prove: if G is abelian then G/H is abelian.
   (2) Prove: if G is cyclic then G/H is cyclic.

6. This exercise will explicitly illustrate the first isomorphism theorem (fundamental homomorphism theorem). Consider a homomorphism $\theta: D_3 \to C_6$ defined as follows (suggested but not required exercise: verify that this is indeed a homomorphism):

   $\theta(e_G) = e_H$ $\quad\theta(R) = e_H$ $\quad\theta(R^2) = e_H$ $\quad\theta(F) = R^3$ $\quad\theta(FR) = R^3$ $\quad\theta(FR^2) = R^3$

   (1) Find and construct an operation table for the image of this homomorphism, $im\ \theta$.
   (2) Find the kernel of this homomorphism, $\ker \theta$.
   (3) Construct and describe the quotient group $D_3/\ker\theta$ (that is, explicitly list the elements and make an operation table).
   (4) Verify that $G/\ker\theta \cong \operatorname{im}(\theta)$. (*Hint*: since these are small finite groups, it may be helpful to compare operation tables.)